Egy diszkrét idejű kamatlábpélda

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Forward interest rate models

Motivation, general problems:

- appropriate 'random field' interest rate models,
- bond price structures,
- no-arbitrage conditions (see Gáll, Pap & Zuijlen [2005a]),
- discrete versus continuous time models (see Gáll, Pap & Zuijlen [2003]),
- (interest rate) derivative pricing, risk management,
- **statistics, model fitting, model selection:** asymptotic theory and empirical tests (see Gáll, Pap & Zuijlen [2004, 2005b, 2006a and 2006b]).

A Heath-Jarrow-Morton forward interest rate models

Heath, Jarrow, Morton [1990], [1992].
The basic object of the market:
$f(t,x)$, forward interest rate at time t, corresponding to maturity $t+x$ $(x, t \in \mathbb{R}_+)$, i.e. the interest rate 'expected by the market' at time t holding from $t+x$,

$$df_t(t,x) = \alpha(t,x)\,dt + \sigma(t,x)\,dW(t),$$

where $\{W(t)\}_{t\in\mathbb{R}_+}$ is a standard Wiener process, i.e.

$$f(t,x) = f(0,x) + \int_0^t \alpha(u,x)\,du + \int_0^t \sigma(u,x)\,dW(u)$$

In particular,
$r(t) := f(t,0)$: spot interest rate at time t.

Bond prices

$P(t, s)$: the price of a zero coupon bond with maturity s at time t,

$$P(t, s) := \exp\left\{-\int_0^{s-t} f(t, u)\, du\right\}, \qquad 0 \leq t \leq s.$$

Remarks:

- continuous compounding convention,
- a single driving process ('random shock') for all maturity.

Forward rate models driven by random fields

- Kennedy [1994], Goldstein [2000]: Gauss driving fields (Brownian sheets),
- Santa-Clara & Sornette [2001]: $\{Z(t,s)\}_{t,s\in\mathbb{R}_+}$ is a random field,
  x: time to maturity,
  for each fixed $x \in \mathbb{R}_+$ we have

$$d_t f(t,x) = \alpha(t,x)\,dt + \sigma(t,x)\,Z(dt,x),$$

where $\{Z(t,s)\}_{t\in\mathbb{R}_+}$ is a martingale ($s \geq 0$), to put in other way,

$$f(t,x) = f(0,x) + \int_0^t \alpha(u,x)\,du + \int_0^t \sigma(u,x)\,Z(du,x).$$

Discrete time case

A discrete model (Gáll, Pap, Zuijlen):

- $(\Omega, \mathcal{F}, \mathbb{P})$: a probability space,
- $\{\mathcal{F}_k\}_{k \in \mathbb{Z}_+}$: a filtration on it,
- $\{S_{k,\ell}\}_{k,\ell \in \mathbb{Z}_+}$: a random field, i.e. $S_{k,\ell}$ is a r.v. for all $k, \ell \in \mathbb{Z}_+$,
- Notation: $\Delta_1 S_{k,\ell} := S_{k+1,\ell} - S_{k,\ell}$.

Interest rates

$f_{k,j}$: **forward interest rate** at time k, with maturity $k+j$, i.e. for the period

$$[k+j \ , \ k+j+1),$$

j: time *to* maturity,

$$f_{k+1,j} = f_{k,j} + \alpha_{k,j} + \beta_{k,j}\Delta_1 S_{k,j},$$

where $\sigma(\alpha_{k,j}), \sigma(\beta_{k,j}) \in \mathcal{F}_k$, $k \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, or equivalently

$$f_{k,j} = f_{0,j} + \sum_{i=0}^{k-1} \alpha_{i,j} + \sum_{i=0}^{k-1} \beta_{i,j}\Delta_1 S_{i,j}.$$

r_k: the spot interest rate for the period $[k, k+1]$,

$$r_k := f_{k,0} \qquad \forall \quad k \in \mathbb{Z}_+.$$

(Zero coupon) Bond prices:

continuous compounding convention,
$P_{k,\ell}$: price of the bond with maturity ℓ at time k,
$P_{k,k} := 1$ (zero coupon) and

$$P_{k,\ell+1} := P_{k,\ell} \exp\left\{-f_{k,\ell-k}\right\},$$

i.e.

$$P_{k,\ell} = \exp\left\{-\sum_{j=0}^{\ell-k-1} f_{k,j}\right\}, \qquad 0 \le k \le \ell.$$

Discounted bond prices

discounting to time $t = 0$:

$$\frac{1}{\prod_{j=0}^{k-1} \mathbf{e}^{r_j}} P_{k,\ell} = \exp\left\{ -\sum_{j=0}^{k-1} r_j \right\} P_{k,\ell},$$

hence the discount factor is

$$D_k = \exp\left\{ -\sum_{j=0}^{k-1} r_j \right\},$$

i.e.

$$\frac{D_{k+1}}{D_k} = \exp\left\{ -r_k \right\}.$$

Stochastic market discount factor

M_k (corresponding to time k)

$M_0 := 1$,

$$\frac{M_{k+1}}{M_k} = \frac{\exp\left\{-r_k + \sum_{j=0}^{\infty} \phi_{k,j} \Delta_1 S_{k,j}\right\}}{\mathbb{E}\left(\exp\left\{\sum_{j=0}^{\infty} \phi_{k,j} \Delta_1 S_{k,j}\right\} \middle| \mathcal{F}_k\right)},$$

where

$$\phi_{k,j}\text{'s}$$

are the **market price of risk** factors, $\mathcal{F}_k$-measurable r.v., $k, j \geq 0$, further we suppose that

$$\sum_{j=0}^{\infty} \phi_{k,j} \Delta_1 S_{k,j}$$

Absence of arbitrage

Theorem.
$M_k P_{k,\ell}$ $\mathbb{P}$-martingale $(\forall \ell) \iff$ there is no arbitrage $(\exists \mathbb{P}^*)$.
Consider ('the pricing kernel') $\frac{d\mathbb{P}^*_K}{d\mathbb{P}_K} = \Lambda_K$, where $\mathbb{P}_K = \mathbb{P}|_{\mathcal{F}_K}$,
$\Lambda_0 := 1$ and

$$\Lambda_{K+1} := \frac{\exp\left\{\sum_{k=0}^{K}\sum_{i=0}^{\infty}\phi_{k,i}\Delta_1 S_{k,i}\right\}}{\prod_{k=0}^{K}\mathbb{E}\left(\exp\left\{\sum_{i=0}^{\infty}\phi_{k,i}\Delta_1 S_{k,i}\right\}\middle|\mathcal{F}_k\right)}.$$

Furthermore $\{\mathbb{P}^*_k\}_{k\in\mathbb{Z}_+}$ are compatible, $\exists \mathbb{P}^*$, and $\mathbb{P}^*_K = \mathbb{P}^*|_{\mathcal{F}_K}$.
Corollary: see the drift conditions given in Gáll, Pap & Zuijlen [2005a].

Theorem.

(general) ‘drift condition’)

The market is free of arbitrage if and only if for all $0 \leq k < \ell$ we have

$$\frac{\mathbb{E}\left(\exp\left\{\sum_{j=0}^{\infty}\psi_\ell(k,j)\Delta_1 S_{k,j}\right\}\Big|\mathcal{F}_k\right)}{\mathbb{E}\left(\exp\left\{\sum_{j=0}^{\infty}\phi_{k,j}\Delta_1 S_{k,j}\right\}\Big|\mathcal{F}_k\right)}$$

$$= \exp\left\{r_k - f_{k,\ell-k-1} + \sum_{j=0}^{\ell-k-2}\alpha_{k,j}\right\},$$

where

$$\psi_\ell(k,j) := \begin{cases} \phi_{k,j} - \beta_{k,j}, & \text{if} \quad 0 \leq j \leq \ell - k - 2 \\ \phi_{k,j}, & \text{if} \quad \ell - k - 1 \leq j. \end{cases}$$

Theorem. ('drift condition' for normal case) Suppose that the vector $(\Delta_1 S_{k,0}, \Delta_1 S_{k,1}, \dots, \Delta_1 S_{k,j})$ is **normally distributed** w.r.t. $\mathbb{P}$ and $\mathbb{P}$-independent of $\mathcal{F}_k$ for all $k, j \in \mathbb{Z}_+$. Let $\phi_{k,j}$ deterministic, $k, j \in \mathbb{Z}_+$, and $\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} |\phi_{k,j_1} \phi_{k,j_2} c(k, j_1, j_2)| < \infty$. Then the no-arbitrage condition implies

$$f_{k,m} = f_{0,m+k} + \sum_{i=0}^{k-1} a_{i,m+k-i-1} + \sum_{i=0}^{k-1} \beta_{i,m+k-i-1} \Delta_1 S_{i,m+k-i-1},$$

where

$$a_{i,\ell} = \beta_{i,\ell} \left[-\sum_{j=0}^{\infty} \phi(i,j) c(j,\ell) + \sum_{j=0}^{\ell-1} \beta_{i,j} c(j,\ell) + \frac{1}{2} c(\ell,\ell) \beta_{i,\ell} \right],$$

and $c(\ell_1, \ell_2) := \operatorname{cov}(\Delta_1 S_{i,\ell_1}, \Delta_1 S_{i,\ell_2})$.

Examples for the driving process

Define the partial sums of the r.v.'s $\eta_{k,\ell}$, i.e.

$$S_{k,\ell} := \sum_{i=0}^{k} \sum_{j=0}^{\ell} \eta_{i,j} \qquad k, \ell \in \mathbb{Z}_+.$$

Then

$$\begin{aligned} S_{k+1,\ell+1} - S_{k,\ell+1} \\ = S_{k+1,\ell} - S_{k,\ell} + \eta_{k+1,\ell+1}. \end{aligned}$$

E.g. $\eta_{k,\ell} \sim \mathcal{N}(0,1)$ ('Gaussian lattice').
Remark: $\{S_{k,\ell}\}_{k\in\mathbb{Z}_+}$ is a *martingale* w.r.t. $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$ for all $\ell \in \mathbb{Z}_+$.

Geometric spatial AR model

Let $\rho \in \mathbb{R}$ be a constant and

$$S_{k,\ell} = \sum_{i=0}^{k} \sum_{j=0}^{\ell} \rho^{\ell-j} \eta_{i,j}, \qquad k, \ell \in \mathbb{Z}_+$$

the driving field, where $\eta_{i,j} \sim \mathcal{N}(0,1)$ are i.i.d. r.v.'s.
Hence

$$S_{k+1,\ell+1} - S_{k,\ell+1}$$
$$= \rho S_{k+1,\ell} - \rho S_{k,\ell} + \eta_{k+1,\ell+1},$$

$k, \ell \in \mathbb{Z}_+$. Then

$$\Delta_1 S_{k,\ell+1} = \rho \Delta_1 S_{k,\ell} + \eta_{k+1,\ell+1},$$

i.e. $\{\Delta_1 S_{k,\ell}\}_{\ell \in \mathbb{Z}_+}$ is an AR(1) process with coefficient ρ.
Remark: $\{S_{k,\ell}\}_{k \in \mathbb{Z}_+}$ is a *martingale* w.r.t. $\{\mathcal{F}_k\}_{k \in \mathbb{Z}_+}$ for all $\ell \in \mathbb{Z}_+$.

No arbitrage in AR models

Theorem. The ‘drift condition’ is of the form:

$$f_{k,\ell+1} = f_{k,\ell} + \alpha_{k,\ell} - \frac{1}{2}\beta_{k,\ell}^2 c(\ell,\ell) - \beta_{k,\ell}\sum_{j=0}^{\ell-1}\beta_{k,j}c(\ell,j) + \beta_{k,\ell}\sum_{j=0}^{\infty}\phi_{k,j}c(\ell,j),$$

where $c(\ell_1,\ell_2) := \operatorname{cov}(\Delta_1 S_{i,\ell_1}, \Delta_1 S_{i,\ell_2})$ and hence

$$f_{k+1,\ell} - f_{k,\ell+1} = \frac{1}{2}\beta_{k,\ell}^2 c(\ell,\ell) + \beta_{k,\ell}\sum_{j=0}^{\ell-1}\beta_{k,j}c(\ell,j) - \beta_{k,\ell}\sum_{j=0}^{\infty}\phi_{k,j}c(\ell,j) + \beta_{k,\ell}\Delta_1 S_{k,\ell}.$$

Corollary.

In case of spatial AR model

, with *market price of risk* structure

$$\phi_{k,j} = \begin{cases} b_j & j \le J \\ 0 & otherwise, \end{cases}$$

and with *volatility* $\beta_{k,\ell} = \beta$ the no/arbitrage criterion implies

$$f_{k,\ell} - f_{k-1,\ell+1} - \varrho(f_{k,\ell-1} - f_{k-1,\ell}) = \beta\eta_{k,\ell} + \frac{\beta^2}{2}\sum_{i=0}^{2\ell}\varrho^i - \beta\sum_{j=\ell}^{J} b_j\varrho^{j-\ell},$$

and hence for all $k \ge 1$, $\ell \ge 1$

$$f_{k,\ell-1} - f_{k-1,\ell} = \beta\sum_{i=0}^{\ell-1}\varrho^{\ell-i-1}\eta_{k,i} + \frac{\beta^2}{2}\left(\sum_{i=0}^{\ell-1}\varrho^i\right)^2$$

$$\beta\sum^{J} \ldots \sum^{j\vee(\ell-1)} \ldots \varrho^{\ell+i-1-2i}$$

Hence

$$f_{k,\ell} - f_{0,k+\ell} - \varrho(f_{k,\ell-1} - f_{0,k+\ell-1})$$

$$= \beta \sum_{j=1}^{k} \eta_{j,k+\ell-j} + \frac{\beta^2}{2} \sum_{j=1}^{k} \sum_{i=0}^{2(k+\ell-j)} \varrho^i - \beta \sum_{j=0}^{J} b_j q_{j,k,\ell}$$

for $k \geq 1$, $\ell \geq 1$, where

$$q_{j,k,\ell} = \begin{cases} \sum_{n=0 \vee (j-k-\ell+1)}^{j-\ell} \varrho^n, & j \geq \ell \\ 0, & \text{otherwise.} \end{cases}$$

Maximum Likelihood Estimators in AR models

- parameters: volatility (β), AR parameter (ϱ), and market price of risk parameters (in $\phi_{k,\ell}$),
- complicated likelihood function,
- non-independent, non-identically distributed sample $\{f_{k,\ell} \mid 1 \leq k \leq K_n, 0 \leq \ell \leq L_n\}$
- sample size: $K_n \times L_n$ with $K_n = nK + o(n)$, $L_n = nK + o(n)$.

Market price of risk structures

1. **Martingale case:** $\phi_{k,\ell} = 0$ for all k, ℓ,
2. **Model A:** take $\boldsymbol{b} = (b_0, b_1, \dots, b_J) \in \mathbb{R}^{J+1}$ and define

$$\phi_{k,j} = \begin{cases} b_j & j \leq J \\ 0 & otherwise. \end{cases}$$

3. **Model B:** take $b \in \mathbb{R}$, $|q| < 1$, $|q\varrho| < 1$, and define

$$\phi_{k,\ell} = \beta \cdot b \cdot q^j,$$

Joint ML estimation of the parameters

Theorem [Spatial geom. AR driving field] The form of market price of risk:

$$\phi_{k,j} = \begin{cases} b_j & j \leq J \\ 0 & \text{otherwise.} \end{cases}$$

Let $H \subset \mathbb{R}^{J+3}$ be a compact set such that for all $(\beta, \varrho, \mathbf{b}) \in H$ we have $\beta > 0$, $\varrho \in (-1, 1)$. Let $(\beta_0,\ \varrho_0,\ \mathbf{b}_0)$ denote the true parameters, where $\mathbf{b}_0 = (b_{0,0},\ b_{0,1},\ldots,\ b_{0,J})$ and we suppose that $(\beta_0,\ \varrho_0,\ \mathbf{b}_0)$ is in the interior of H. For each $n \in \mathbb{N}$, let $(\widehat{\beta}_n, \widehat{\varrho}_n, \widehat{\mathbf{b}}_n)$ denote a maximum likelihood estimation of $(\beta_0, \varrho_0, \mathbf{b}_0)$, which maximises the loglikelihood $\mathcal{L}_{K_n,L_n}$ over H, and
$\widehat{\mathbf{b}}_n = (\widehat{b}_{n,0}, \widehat{b}_{n,1}, \ldots, \widehat{b}_{n,J})$.

- Then the ML estimation of $(\beta_0, \varrho_0, \mathbf{b_0})$ is strongly consistent, that is

$$(\widehat{\beta}_n, \widehat{\varrho}_n, \widehat{\mathbf{b}}_n) \to (\beta_0, \varrho_0, \mathbf{b}_0) \qquad \text{m.b.}, \ n \to \infty.$$

- Furthermore

$$\begin{bmatrix} n(\widehat{\beta}_n - \beta_0) \\ n(\widehat{\varrho}_n - \varrho_0) \\ \sqrt{n}(\widehat{\mathbf{b}}_n - \mathbf{b}_0) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Lambda), \qquad n \to \infty,$$

such that Λ is of the form

$$\Lambda = \begin{bmatrix} \Lambda_1 & \mathbf{0} \\ \mathbf{0} & \Lambda_2 \end{bmatrix},$$

where

$$\Lambda_1 = \left(\sigma_{1,1}\sigma_{2,2} - \sigma_{1,2}^2\right)^{-1} \begin{bmatrix} \sigma_{2,2} & -\sigma_{1,2} \\ -\sigma_{1,2} & \sigma_{1,1} \end{bmatrix}$$

and

$$\sigma_{1,1} = \frac{2KL}{\beta_0^2} + \frac{K(K+2L)}{2\left(1-\varrho_0\right)^2},$$

$$\sigma_{2,2} = \frac{KL}{1-\varrho_0^2} + \frac{K(K+2L)\beta_0^2}{2\left(1-\varrho_0\right)^4},$$

$$\sigma_{1,2} = \sigma_{2,1} = \frac{K(K+2L)\beta_0}{2\left(1-\varrho_0\right)^3},$$

furthermore, Λ_2 is a matrix of size $(J+1) \times (J+1)$ such that

$$\Lambda_2 = \frac{1}{K} \begin{bmatrix} 1+\varrho_0^2 & -\varrho_0 & 0 & 0 & 0 & \dots & 0 \\ -\varrho_0 & 1+\varrho_0^2 & -\varrho_0 & 0 & 0 & \dots & 0 \\ 0 & -\varrho_0 & 1+\varrho_0^2 & -\varrho_0 & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & 0 & -\varrho_0 & 1+\varrho_0^2 & -\varrho_0 \\ 0 & 0 & \dots & \dots & 0 & -\varrho_0 & 1 \end{bmatrix}.$$

Empirical problems

- joint estimation using numerical methods, CI's,
- speed of convergence,
- models selection: choice of model and # market price of risk parameters, i.e. J, (LL, AIC, etc),
- numerical questions: initial values, initial yield curve,
- effect on risk: a VaR example,
- Note: some 'early' test results, given 'small' sample size.

Recent methods, motivation:

- 'calibration' methods under an EMM,
- overfitting,
- underlying versus derivative data,
- 'short & long term' interest rate behaviour (market price structure),
- measuring risk (under the market measure).

Joint estimation

	$(\rho,\beta)=(1,1)$				$(\rho,\beta)=(0.7,1)$			
	ρ		β^2		ρ		β^2	
(K,L)	AMD $*10^3$	SE $*10^5$	AMD $*10^2$	SE $*10^3$	AMD $*10^1$	SE $*10^3$	AMD	SE $*10^2$
(10,3)	1591	2023	736.4	96.97	68.70	61.12	13.59	17 .15
(20,6)	252.0	300.8	211.2	25.57	31.34	29.01	7.342	10 .12
(30,9)	73.45	88.57	89.73	10.68	22.33	19.26	5.609	6. 687
(40,12)	31.35	43.14	50.25	6.783	15.65	13.12	4.091	5. 032
(50,15)	16.08	18.81	31.38	3.667	12.58	10.40	3.285	4. 080
(80,24)	3.089	5.075	8.897	1.541	8.407	7.420	2.246	2. 790

Average mean difference (AMD) and its standard error (SE) for ρ and β. Note that for the sake of simplicity of the table the values of AMD and SE are multiplied by some factors which are shown in line 4.

Volatility estimation

(K,L)	$b=1$ AMD	$b=1$ SE	$b=0$ AMD	$b=0$ SE	$\leq$ 1%	$\leq$ 0.1%	$\leq$ 0.01%
(10,3)	2.51	4.28	1.56	7.02	(2,2)	(5,7)	(15,19)
(20,6)	$4.84 \cdot 10^{-1}$	$3.41 \cdot 10^{-1}$	$3.91 \cdot 10^{-1}$	$4.91 \cdot 10^{-1}$	(1,1)	(3,4)	(8,9)
(30,9)	$1.98 \cdot 10^{-1}$	$6.50 \cdot 10^{-2}$	$1.94 \cdot 10^{-1}$	$8.42 \cdot 10^{-2}$	(1,1)	(2,2)	(5,6)
(40,12)	$1.07 \cdot 10^{-1}$	$2.41 \cdot 10^{-2}$	$1.05 \cdot 10^{-1}$	$2.84 \cdot 10^{-2}$	(1,1)	(2,2)	(4,4)
(50,15)	$7.08 \cdot 10^{-2}$	$8.36 \cdot 10^{-3}$	$6.04 \cdot 10^{-2}$	$9.59 \cdot 10^{-3}$	(1,1)	(1,1)	(3,3)
(100,30)	$1.73 \cdot 10^{-2}$	$6.16 \cdot 10^{-4}$	$1.70 \cdot 10^{-2}$	$6.59 \cdot 10^{-4}$	(1,1)	(1,1)	(2,2)

Column 1: sample size. Column 2-5: the average mean difference (AMD) and its estimated standard error (SE) for β. Column 6-8: the average minimal value of n needed for the average mean difference for β to be below a certain threshold when $K_n := nK$ and $L_n := nL$ (first value) and $L_n = L$ (second value).

Model selection, Model A, unknown J

	AIC		AICc		LLF	
(K, L)	$\mathbb{P}$	mean	$\mathbb{P}$	mean	$\mathbb{P}$	mean
(10,3)	0.375	1.455	0.435	1.290	0.265	3.085
(20,6)	0.655	1.850	0.665	1.820	0.255	3.245
(30,9)	0.740	2.090	0.735	2.075	0.270	3.165
(40,12)	0.810	2.265	0.815	2.255	0.280	3.220
(50,15)	0.795	2.250	0.795	2.245	0.295	3.250
(80,24)	0.850	2.250	0.850	2.250	0.250	3.410
(100,30)	0.850	2.245	0.850	2.245	0.180	3.570

$J = 2$ Column 2, 4, 6: the probability that the program will find the right J. Column 3, 5, 7: the average value of J found by the program.

Model selection, Model A and Model B

true model: Model A

(K, L)	AIC	AICc	LLF
(10,3)	65	48	6
(20,6)	70	65	4
(30,9)	81	81	2
(40,12)	79	79	7
(50,15)	76	75	12
(100,30)			

The table shows (for 100 runs) how often Model A has been chosen by AIC, AICc and LLF.

Example: sensitivity of Value at Risk

(changing the values of the parameters)
Zero coupon bond: principal value 1000, terminal date in 32 steps, value in 20 steps.

	Bond price	Relative Value at Risk			
	mean	99.9%	99%	95%	90%
True parameters	690.59	57.103	43.037	30.814	24.649
$\beta = 0.0018$	690.51	51.916	39.052	27.940	21.953
$\varrho = -0.12$	690.35	56.097	42.402	30.330	24.232
$\mathbf{b} = (0.4, 0.4)$	690.01	54.896	42.972	31.137	24.158

$\beta = 0.002$, $\varrho = -0.1$, $J = 2$ and $\mathbf{b} = (0.2, 0.2)$

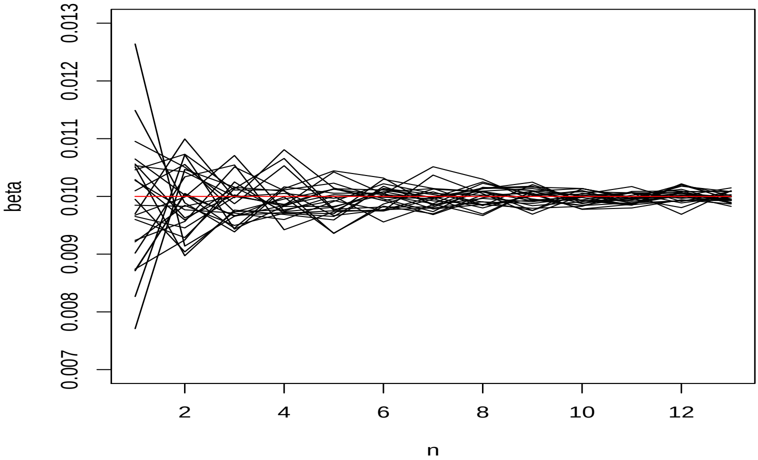
Example: sensitivity of Value at Risk, (model selection)

Zero coupon bond: principal value 1000, terminal date in 32 steps, value in 20 steps.

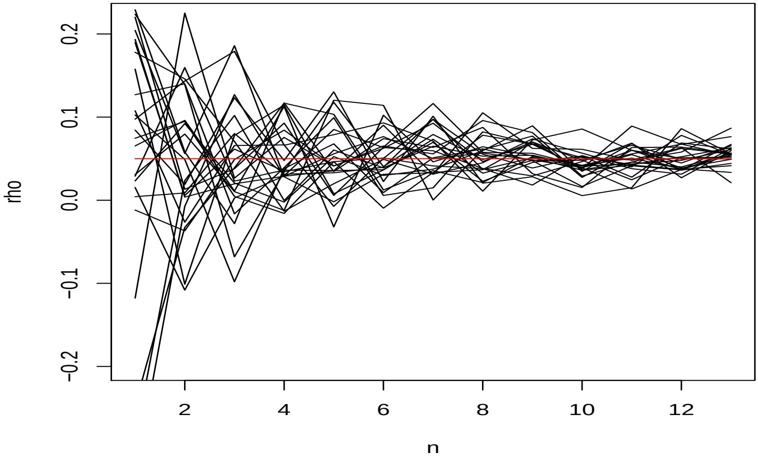
	Bond price	Relative Value at Risk			
	mean	99.9%	99%	95%	90%
True parameters	692.47	55.932	42.693	30.664	24.224
$J = 1$	692.83	57.217	42.722	30.116	23.570
$J = 2$	692.94	54.522	42.967	30.798	24.287

$\beta = 0.002$, $\varrho = -0.1$, $J = 2$ and $\mathbf{b} = (0.2, 0.2)$

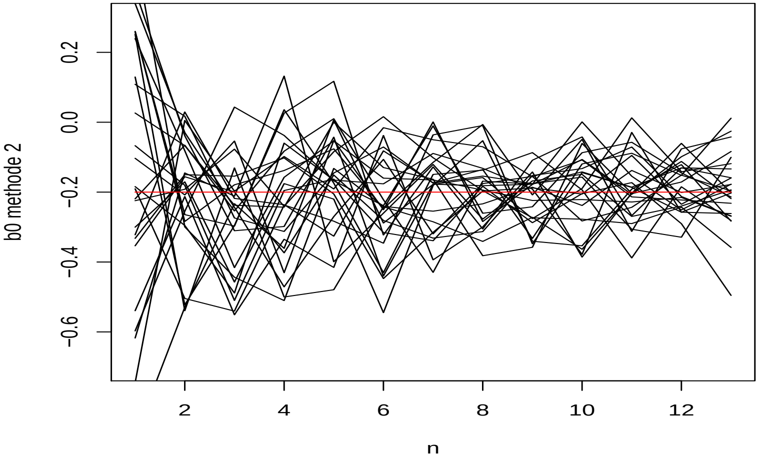
Speed of $\widehat{\beta}$



Speed of $\hat{\varrho}$



Speed of $\widehat{b}_0$ (J=2)



Conclusions on the numerical results

- $\widehat{\beta}$ is fairly good even for relatively small sample size,
- one needs larger sample size, however, for $\widehat{\varrho}$ and in particular for $\widehat{\boldsymbol{b}}$,
- method 2 is suggested for $\widehat{\boldsymbol{b}}$,
- AIC is fairly good (promising) for chosing the right model, either in the class of Model A (even for small sample size), or in the class of Model A and Model B,
- though further study is needed,
- the right market price of risk structure is important to measure the risk (e.g. VaR).

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